

# Subsonic-supersonic condition for shocks

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The necessary conditions for stability of a shock, that the shock travel with supersonic velocity with respect to the medium ahead and with subsonic velocity with respect to the medium behind, are shown to be implied by the second law of thermodynamics for very general classes of viscous, heat-conducting fluids; the Weyl conditions are not invoked. The results are shown to be also compatible with the Le Chatelier-Braun principle. They further imply that under certain conditions it is not thermodynamically permissible to assume the existence of a shock transition layer in which entropy production is due to heat conduction alone.

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## I. INTRODUCTION

In studies of the structure and stability of shock waves it is usually demonstrated or assumed that a necessary condition for a stable shock to exist is that the shock travel with supersonic velocity with respect to the medium ahead of the wave and with subsonic velocity with respect to the medium behind.<sup>1-3</sup> These restrictions can be expressed by the inequalities

$$M_0 \geq 1, \quad M_1 \leq 1, \quad (1)$$

where

$$M = |v/c|$$

is the Mach number, given by the magnitude of the ratio of the flow velocity to the local sound speed in a coordinate system in which the shock front is stationary. Subscripts "0" refer to the initial state ahead of the shock front and subscripts "1" to the final state behind.

For restricted classes of materials conditions (1) have been shown to follow from the second law of thermodynamics. Thus, Weyl proved (1) for materials subject to the conditions

$$(\partial^2 P / \partial V^2)_S > 0, \quad (\partial P / \partial S)_V > 0. \quad (2)$$

Landau and Lifshitz also show that (2) implies (1) for compressive shocks in both the weak shock approximation, in which no distinction is made between Hugoniot and isentropic pressure-volume curves, and for shocks of arbitrary strength (Ref. 4, p. 323). Morduchow and Libby have derived (1) for an ideal gas.<sup>5</sup> Cowperthwaite considered a case in which (2) is violated in the neighborhood of the initial state and showed that a postulated compressive shock with final state in this region is impossible because it would result in a net entropy decrease.<sup>6</sup> This result is consistent with stability arguments based on acoustic wave interactions as described by Duvall.

For rarefaction shocks the arguments are reversed so that (1) is valid when  $(\partial^2 P / \partial V^2)_S < 0$ .

In this paper we show that inequalities (1) are a necessary consequence of the second law of thermodynamics for very general classes of fluids. In particular, we do not assume (2), but only the well-known thermodynamic stability criteria,

$$(\partial P / \partial V)_S < (\partial P / \partial V)_T < 0 \quad (3a)$$

and

$$C_p = T(\partial S / \partial T)_p > C_v = T(\partial S / \partial T)_v > 0. \quad (3b)$$

We also assume the existence of an equilibrium surface, or fundamental equation, representing the locus of reversible paths joining equilibrium states. This surface is characterized by the usual relation,

$$S = S(E, \bar{V}, N), \quad (4)$$

where  $S$  is entropy,  $E$  is internal energy,  $N$  is mole-number, and  $\bar{V}$  is volume. If we denote specific quantities per unit mass by  $s$ ,  $e$ , and  $V$  and consider only closed systems, then (4) is equivalent to

$$s = s(e, V)$$

which is assumed to be invertible to give  $e = e(V, s)$ . The equilibrium pressure and temperature are defined by

$$P = P(e, V) \equiv -(\partial e / \partial V)_s, \quad (5)$$

$$T = T(e, V) \equiv (\partial e / \partial s)_V.$$

We place no restriction on the amplitude of the shock, nor do we assume an explicit form for the constitutive equations relating the viscous dissipation and the heat conduction to velocity and temperature gradients.

## II. THERMODYNAMIC PRINCIPLES

Truesdell has given a general formulation of the first and second laws for irreversible processes.<sup>7</sup> Following his development we write the first law in the form,

$$\rho \dot{e} = w + \text{div } \mathbf{h}, \quad (6)$$

where  $\rho$  is density,  $\dot{e}$  is the time rate of change of the specific internal energy of a mass element,  $w$  is the rate at which mechanical work is performed on an infinitesimal volume containing the mass element, and  $\mathbf{h}$  is the negative of the heating flux vector. The heating influx is therefore,

$$q = \mathbf{h} \cdot \mathbf{n},$$

where  $\mathbf{n}$  is the outward unit normal to the surface of the volume.

The internal dissipation is defined by the relation

$$\delta = T \dot{s} - V \text{div } \mathbf{h}. \quad (7)$$

It is thus given by the difference between the rate of entropy increase, multiplied by temperature, and the rate of energy increase due to heat conduction.

For reversible processes Gibbs' relation applies

$$\dot{e} = T\dot{s} - P\dot{V}. \quad (8)$$

Hence, combining (7) and (8),

$$\delta = \dot{e} + P\dot{V} - V \operatorname{div} h$$

and, employing (6),

$$\rho\delta = w + \rho P\dot{V}. \quad (9)$$

We now specialize to one-dimensional flow and denote the total stress acting in the direction of the flow by  $\sigma$ , with compressive stress measured positive. Then,

$$w = -\rho\sigma\dot{V}$$

and, therefore

$$\delta = -(\sigma - P)\dot{V}. \quad (10)$$

Note that in this derivation we have not necessarily assumed that  $P$ ,  $T$ , or  $s$ , are given by their local current values, which have not been defined, but are given instead by their values on the associated equilibrium surface. The real, nonequilibrium path followed by a material element is thus mapped onto the equilibrium surface at corresponding values of  $e$  and  $V$ . These latter quantities, of course, are well-defined whether or not equilibrium obtains. In effect, we observe the "shadow" of the real process on the equilibrium surface.

From this point of view it is not obvious that the second law need always apply in terms of the quantities thus defined. For small deviations from equilibrium the relation (4) still holds, however, and the Second Law is also assumed to be valid. We shall restrict our attention to such small deviations in the following. This point of view is discussed also by Landau and Lifshitz (Ref. 4, p. 187).

The second law, requiring that entropy production be positive, is expressed by the Clausius-Duhem inequality,

$$\rho\delta - \mathbf{h} \cdot \boldsymbol{\gamma} \geq 0, \quad (11)$$

where  $\boldsymbol{\gamma}$  is defined as

$$\begin{aligned} \boldsymbol{\gamma} &= (-1/T)\operatorname{grad} T \\ &= (-1/T)(\partial T/\partial x). \end{aligned} \quad (12)$$

### III. APPLICATION TO SHOCKS

We now apply these principles, expressed by relations (6)–(12), to a steady wave of "permanent regime," or shock wave. The shock transition region joins initial and final states which are assumed to be in internal thermodynamic equilibrium; it is depicted in a coordinate system in which the front is stationary in Fig. 1.

Since the wave is stationary and one-dimensional, we can write the relation between material and spatial time derivatives as

$$d/dt = \partial/\partial t + v \partial/\partial x = v \partial/\partial x. \quad (13)$$

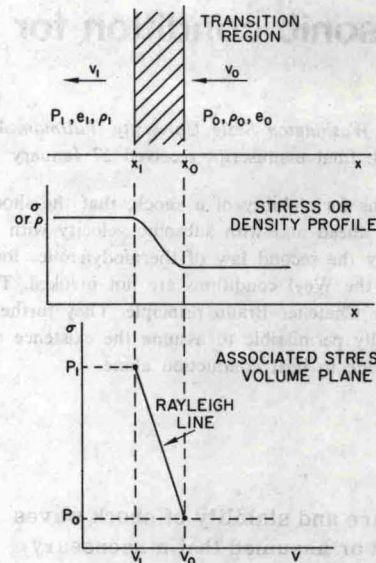


FIG. 1. Shock transition layer, compressive shock.

Consequently, (12) can be expressed as,

$$\begin{aligned} \boldsymbol{\gamma} &= (-1/T)(\partial T/\partial x) = -(1/Tv)(dT/dt) \\ &= -T^{-1}v^{-1}(dT/dV)\dot{V}, \end{aligned} \quad (14)$$

where  $dT/dV$  is the directional derivative in the equilibrium surface of temperature with respect to volume along the projection of the path followed by a material element as it traverses the shock front, and  $\dot{V}$  is the material time derivative,  $\dot{V} = dV/dt$ .

Application of the Clausius-Duhem inequality, (11) also requires expressions for  $\delta$  and for  $\mathbf{h}$ . From (7), (10), and (13) we have

$$\begin{aligned} \operatorname{div} h &= \partial h/\partial x = \rho T\dot{s} - \rho\delta \\ &= \rho[T\dot{s} + (\sigma - P)\dot{V}] \end{aligned} \quad (15)$$

or

$$dh = \rho v [T ds/dV + (\sigma - P)dV]. \quad (16)$$

To first-order terms in a series expansion about an equilibrium state, therefore,

$$h(V) - h(V_r) = \rho v T (ds/dV)(V - V_r)$$

or, since the heat flux vector is zero in an equilibrium state,  $V_r$ ,

$$h(V) = \rho v T ds/(ds/dV)(V - V_r). \quad (17)$$

The mechanical dissipation can also be approximated by the first term of a series expansion. Thus,

$$\begin{aligned} \delta &= -(\sigma - P)\dot{V}, \\ \delta(V) &= -[d\sigma/dV - dP/dV]_{V_r} (V - V_r)\dot{V}. \end{aligned} \quad (18)$$

We now note that the path in the  $\sigma - V$  plane followed by a mass element is represented by the straight line joining the equilibrium end states (Rayleigh line.)<sup>8</sup> Its slope is given by

$$-j^2 = (P_1 - P_0)/(V_1 - V_0) = (\sigma - P_0)/(V - V_0), \quad (19)$$

where  $j$  is the mass flux,